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Partition-complete spaces are preserved by tri-quotient maps

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Abstract

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The result stated in the title is proved.

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1. Introduction

A number of topological completeness properties, all equivalent to complete metrizability in metrizable spaces, have been studied in the literature. These include, in order of introduction, Čech-completeness [2], sieve-completeness (=monotonic Čech-completeness) [3, 6], and partition-completeness [7, 11, 9]. A major reason for considering these progressively more general concepts is that they tend to be preserved by progressively larger classes of maps. Specifically, Čech-completeness is preserved by perfect maps (provided the range is completely regular) but not by open maps [4, p. 199], while sieve-completeness is preserved by both perfect maps and open maps [3, 6] and, more generally, by tri-quotient maps [6, Theorem 6.3]. Partition-completeness was shown to be preserved by perfect maps and by open maps in [9, Theorems 11 and 12]; the principal purpose of this paper is to show that, more generally, it is also preserved by tri-quotient maps.

Background material on complete sequences of covers and complete sieves is reviewed in Section 2. Section 3 deals with exhaustive covers and partition-complete spaces, and obtains some general conditions for a map to have a partition-complete range. Section 4 establishes some technical lemmas about tri-quotient maps, and in Section 5 these lemmas are combined with the results of Section 3 to prove, in Corollary 5.2, that tri-quotient maps preserve partition-completeness.

All *maps* in this paper are continuous, and no separation properties are assumed.

2. Complete sequences of covers and complete sieves

A sequence $(\mathscr{U}_n)$ of covers of a space X is *complete* if, whenever $U_n \in \mathscr{U}_n$ for all n, then every filter base $\mathscr{F}$ on X which is controlled[1] by (U_n) clusters in X. It was shown in [1, 5] that a completely regular space X is Čech-complete (i.e., a G_δ in βX) if and only it has a complete sequence of open covers.

A *sieve* on a space X is a sequence of indexed covers[2] $\{U_\alpha: \alpha \in A_n\}$ $(n \geq 0)$ of X, together with functions $\pi_n : A_{n+1} \to A_n$, such that $U_\alpha = X$ for $\alpha \in A_0$ and $U_\alpha = \bigcup \{U_\beta : \beta \in \pi_n^{-1}(\alpha)\}$ for al $\alpha \in A_n$ and all n. Such a sieve is called *complete* if, whenever $\alpha_n \in A_n$ with $\pi_n(\alpha_{n+1}) = \alpha_n$ for all n, then every filter base on X which is controlled by (U_{α_n}) clusters in X. A sieve $(\{U_\alpha : \alpha \in A_n\}, \pi_n)$ on X is called *open* if every U_α is open in X. A space X is called *sieve-complete* [6] (=monotonically Čech-complete [2]) if it has a complete open sieve.[3] Every space with a complete sequence of open covers is sieve-complete, and the converse is true in paracompact spaces [3, Remark 3.9; 6, Theorem 3.2].

3. Exhaustive covers and partition-complete spaces

A cover $\mathscr{U}$ of a space X is *exhaustive* [7] if every nonempty $S \subset X$ has a nonempty, relatively open subset of the form $U \cap S$ with $U \in \mathscr{U}$.

Lemma 3.1 [7, Lemma 2.1]. *The following are equivalent for an indexed cover $(U_\alpha)_{\alpha \in A}$ of a space X.*

(a) *$(U_\alpha)_{\alpha \in A}$ is an exhaustive cover of X.*

(b) *The index set A can be well-ordered such that $\bigcup_{\alpha' \leq \alpha} U_{\alpha'}$ is open in X for every $\alpha \in A$.*

A sieve $(\{U_\alpha : \alpha \in A_n\}, \pi_n)$ is called *exhaustive* if $\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\}$ is an exhaustive cover of U_α for all $\alpha \in A_n$ and all n.

Proposition 3.2 [7, Proposition 4.1]. *The following are equivalent for any space X.*

(a) *X has a complete sequence of exhaustive covers.*

(b) *X has a complete sequence of exhaustive disjoint covers.*

(c) *X has a complete exhaustive sieve.*

A space X satisfying the equivalent conditions of Proposition 3.2 is called *partition-complete* [9].

[1] $\mathscr{F}$ is *controlled* by (U_n) if each U_n contains some $F \in \mathscr{F}$.

[2] The index sets are assumed to be disjoint. Unlike [3, 6], but like [7], we do not assume here that the sets U_α are open in X.

[3] A regular space is sieve-complete if and only if it is a λ_b-space in the sense of [10].

Since open covers are exhaustive, every sieve-complete space is partition-complete. The converse is generally false [7, Corollary 8.3], but it is true in metrizable spaces [7, Theorem 1.5; 8]. A much sharper result was proved by Wicke in [11]: A regular space is sieve-complete if and only if it is partition-complete and a monotone p-space[4].

We now turn to conditions under which the range of a map is partition-complete.

Proposition 3.3. *Let $f: X \to Y$ be a map, and let $(\mathcal{U}_n)$ be a complete sequence of covers of X. Suppose X has a cover $\mathcal{E}$ such that $X \in \mathcal{E}$ and, whenever $E \in \mathcal{E}$ and E is $\mathcal{U}_n$-small*[5] *for some n, then*

$$\{f(E'): E' \in \mathcal{E},\ E' \subset E,\ E' \text{ is } \mathcal{U}_{n+1}\text{-small}\}$$

is an exhaustive cover of $f(E)$. Then Y is partition-complete.

Proof. We may clearly assume that $\{x\} \in \mathcal{E}$ for every $x \in X$, and that $\mathcal{U}_0 = \{X\}$. Let $\mathcal{E}_n = \{E \in \mathcal{E}: E \text{ is } \mathcal{U}_n\text{-small}\}$. By induction, we can easily construct a sieve $(\{E_\alpha : \alpha \in A_n\}, \pi_n)$ on X such that

$$\{E_\beta : \beta \in \pi_n^{-1}(\alpha)\} = \{E' \in \mathcal{E}_{n+1} : E' \subset E_\alpha\}$$

for all $\alpha \in A_n$ and all n. By our assumptions, this implies that $(\{f(E_\alpha): \alpha \in A_n\}, \pi_n)$ is an exhaustive sieve on Y. Moreover, since $\mathcal{E}_n$ refines $\mathcal{U}_n$ for all n, $(\{E_\alpha : \alpha \in A_n\}, \pi_n)$ is a complete sieve on X, so $(\{f(E_\alpha): \alpha \in A_n\}, \pi_n)$ is a complete sieve on Y by [6, Lemma 4.1]. Hence Y satisfies Proposition 3.2(c), so Y is partition-complete. □

Corollary 3.4. *Let $f: X \to Y$ be a map. Suppose X has a cover $\mathcal{E}$ such that $X \in \mathcal{E}$ and, for every $E \in \mathcal{E}$ and every exhaustive cover $\mathcal{U}$ of E,*

$$\{f(E'): E' \in \mathcal{E},\ E' \text{ is } \mathcal{U}\text{-small}\}$$

is an exhaustive cover of $f(E)$. Then, if X is partition-complete, so is Y.

Proof. Let $(\mathcal{U}_n)$ be a complete sequence of exhaustive covers of X. Then $(\mathcal{U}_n)$ and $\mathcal{E}$ satisfy the hypotheses of Proposition 3.3, since $\mathcal{U}_n$ being an exhaustive cover of X implies that $\{U \cap E: U \in \mathcal{U}_n\}$ is an exhaustive cover of E for every $E \subset X$. Hence Y is partition-complete by Proposition 3.3. □

Remark. For a collection $\mathcal{U}$ of sets, let us denote $\{\bigcup \mathcal{F}: \mathcal{F} \subset \mathcal{U},\ \mathcal{F} \text{ finite}\}$ by $\mathcal{U}^f$. It follows from [5, Theorem 2.14] that, if $(\mathcal{U}_n)$ is a complete sequences of covers of X, then so is $(\mathcal{U}_n^f)$.[6] This implies that Proposition 3.3 remains valid with "$\mathcal{U}_n$-small"

[4] Monotone p-spaces are defined in [3, Definition 2.1]. Every p-space—and hence every metrizable space—is a monotone p-space.

[5] E is *$\mathcal{U}$-small* if $E \subset U$ for some $U \in \mathcal{U}$.

[6] It is assumed in [5, Theorem 2.14] that X is regular, but with our slightly different definition of a complete sequence of covers that assumption is not needed.

changed to "$\mathcal{U}_n^f$-small" and with "$\mathcal{U}_{n+1}$-small" changed to "$\mathcal{U}_{n+1}^f$-small", and consequently that Corollary 3.4 can be sharpened by changing "$\mathcal{U}$-small" to "$\mathcal{U}^f$-small". The modified version of Proposition 3.3 obtained in this way generalizes [7, Theorem 1.6].

4. Tri-quotient maps

According to [6, Definition 6.1], a map $f: X \to Y$ is called *tri-quotient* if one can assign to every open $U \subset X$ an open $U^* \subset Y$ satisfying the following conditions:

(a) $U^* \subset f(U)$.

(b) $X^* = Y$.

(c) $U \subset V$ implies $U^* \subset V^*$.

(d) If $y \in U^*$ and $\mathcal{W}$ is a cover of $f^{-1}(y) \cap U$ by open subsets of X, then there is a finite $\mathcal{F} \subset \mathcal{W}$ such that $y \in (\bigcup \mathcal{F})^*$.

We call $U \to U^*$ *a tri-quotiency assignment*, or *t-assignment*, for f.

By [6, Theorem 6.5], tri-quotient maps include all open maps, all perfect (or merely inductively perfect[7]) maps, and all compact-covering maps $f: X \to Y$ with Lindelöf fibers from a regular space X onto a first-countable Hausdorff space Y. Moreover, if X is a regular sieve-complete space and Y is paracompact, then $f: X \to Y$ is tri-quotient if and only if it is inductively perfect [6, Theorem 6.6]. I don't know whether this last result remains true if "sieve-complete" is weakened to "partition-complete".

The following Lemma 4.1 is used in the proof of Lemma 4.2, while Lemmas 4.2 and 4.3 will be used to prove Theorem 5.1.

Lemma 4.1. *Let $f: X \to Y$ be tri-quotient, and let U, R be open in X with $U \cap f^{-1}(U^*) \subset R \subset U \cup f^{-1}(U^*)$. Then $R^* = U^*$.*

Proof. (a) $R^* \subset U^*$: Suppose not. Let $y \in R^* \setminus U^*$. Then $y \in R^*$ and $U \supset f^{-1}(y) \cap R$, so $y \in U^*$ because f is tri-quotient, contradicting $y \in R^* \setminus U^*$.

(b) $U^* \subset R^*$: Suppose not. Let $y \in U^* \setminus R^*$. Then $y \in U^*$ and $R \supset f^{-1}(y) \cap U$, so $y \in R^*$ because f is tri-quotient, contradicting $y \in U^* \setminus R^*$. □

Lemma 4.2. *Let $f: X \to Y$ be tri-quotient, let V and W be open in X, and let $P = W \setminus V$, $Q = W^* \setminus V^*$, and $E = P \cap f^{-1}(Q)$. Then $f(E) = Q$ and $f: E \to Q$ is tri-quotient.*

Proof. Clearly $f(E) \subset Q$. Suppose $f(E) \neq Q$, and let $y \in Q \setminus f(E)$. Then $y \notin f(P)$. Thus $y \in W^*$ and $V \supset f^{-1}(y) \cap W$, so $y \in V^*$ because f is tri-quotient, contradicting $y \in Q = W^* \setminus V^*$. Hence $f(E) = Q$.

[7] A map $f: X \to Y$ is *inductively perfect* if there is an $X' \subset X$ such that $f(X') = Y$ and $f|X'$ is perfect. (If X is Hausdorff, this X' must be closed in X.)

Now let $S = V \cup f^{-1}(V^*)$ and $T = W \cap f^{-1}(W^*)$. Then $E = T \setminus S$. Also $S^* = V^*$ and $T^* = W^*$ by Lemma 4.1, so $Q = T^* \setminus S^*$.

For each relatively open $U \subset E$, define $\tilde{U} = [U \cup (T \cap S)]^* \setminus S^*$, and let us show that $U \to \tilde{U}$ is a t-assignment for $f|E : E \to Q$. Clearly $U \to \tilde{U}$ is order-preserving. Let us show that $\tilde{U} \subset f(U)$. Suppose not, and let $y \in \tilde{U} \setminus f(U)$. Then $y \in [U \cup (T \cap S)]^*$, and $T \cap S \supset f^{-1}(y) \cap [U \cup (T \cap S)]$, so $y \in (T \cap S)^*$ because f is tri-quotient, contradicting $y \notin S^*$ (since $y \in \tilde{U}$). Thus $\tilde{U} \subset f(U)$. In particular, $\tilde{E} \subset f(E) = Q = T^* \setminus S^* = \tilde{E}$, so $\tilde{E} = f(E)$. Finally, it is easy to check that, if $y \in \tilde{U}$ and if $\mathcal{G}$ is a cover of $f^{-1}(y) \cap U$ by relatively open subsets of E, then $y \in (\bigcup \mathcal{F})^{\sim}$ for some finite $\mathcal{F} \subset \mathcal{G}$. Hence $U \to \tilde{U}$ is a t-assignment for $f|E$, so $f|E$ is tri-quotient. □

Lemma 4.3. *Let $f: X \to Y$ be tri-quotient, and let $\mathcal{W}$ be a collection of open subsets of X which is preserved by finite unions. Then $(\bigcup \mathcal{W})^* = \bigcup \{W^* : W \in \mathcal{W}\}$.*

Proof. This follows immediately from the definition of a tri-quotient map. □

5. Tri-quotient maps, exhaustive covers and partition-complete spaces

Theorem 5.1. *Let $f: X \to Y$ be tri-quotient, and let $(U_\alpha)_{\alpha \in A}$ be an exhaustive cover of X. Then there exist $E_\alpha \subset U_\alpha$ such that $(f(E_\alpha))$ is an exhaustive cover of Y and $f|E_\alpha : E_\alpha \to f(E_\alpha)$ is tri-quotient for all $\alpha \in A$.*

Proof. By Lemma 3.1, we can well-order the index set A such that $W_\alpha = \bigcup_{\alpha' \leqslant \alpha} U_{\alpha'}$ is open in X for every $\alpha \in A$. We may suppose that A has a largest element α_0, with $U_{\alpha_0} = \emptyset$, so that $W_{\alpha_0} = X$.

For each $\alpha \in A$, let $V_\alpha = \bigcup_{\alpha' < \alpha} W_{\alpha'}$, let $P_\alpha = W_\alpha \setminus V_\alpha$, let $Q_\alpha = W_\alpha^* \setminus V_\alpha^*$, and let $E_\alpha = P_\alpha \cap f^{-1}(Q_\alpha)$. Note that $E_\alpha \subset P_\alpha \subset U_\alpha$. Also, by Lemma 4.2, we have $f(E_\alpha) = Q_\alpha$ and $f|E_\alpha : E_\alpha \to Q_\alpha$ is tri-quotient for all α. It remains to show that (Q_α) is an exhaustive cover of Y.

By Lemma 4.3, we have $V_\alpha^* = \bigcup_{\alpha' < \alpha} W_{\alpha'}^*$ for all α, so $Q_\alpha = W_\alpha^* \setminus \bigcup_{\alpha' < \alpha} W_{\alpha'}^*$. Thus $\bigcup_{\alpha' \leqslant \alpha} Q_{\alpha'} = W_\alpha^*$, which is open in Y, and $\bigcup_{\alpha' \leqslant \alpha_0} Q_{\alpha'} = W_{\alpha_0}^* = X^* = Y$. Hence (Q_α) is an exhaustive cover of Y by Lemma 3.1. □

Remark. As the proof of Theorem 5.1 shows, the cover $(f(E_\alpha))$ of Y is actually disjoint.

Corollary 5.2. *Let $f: X \to Y$ be tri-quotient. Then, if X is partition-complete, so is Y.*

Proof. Let $\mathcal{E} = \{E \subset X : f|E$ is tri-quotient$\}$. By Theorem 5.1, this $\mathcal{E}$ satisfies the hypotheses of Corollary 3.4, and hence our assertion follows from Corollary 3.4. □

In conclusion, it may be of interest to compare Theorem 5.1 with the following analogue in which all sets are open and whose proof is significantly simpler than that of Theorem 5.1.

Theorem 5.3. *Let $f: X \to Y$ be tri-quotient, and let (U_α) be an open cover of X preserved by finite unions. Then there exist open $E_\alpha \subset U_\alpha$ such that $(f(E_\alpha))$ is an open cover of Y and $f|E_\alpha : E_\alpha \to f(E_\alpha)$ is tri-quotient for all $\alpha \in A$.*

Proof. Simply let $E_\alpha = U_\alpha \cap f^{-1}(U_\alpha^*)$. Clearly E_α is open in X, and $(f(E_\alpha))$ covers Y because (U_α) is preserved by finite unions. It is easy to check that $f(E_\alpha) = U_\alpha^*$ (which is open in Y), and that $f|E_\alpha$ is tri-quotient because $V_\alpha \to V_\alpha^*$, for V_α open in E_α (and hence in X), is a t-assignment for $f|E_\alpha$. $\square$

Remark. The assumption in Theorem 5.3 that (U_α) is preserved by finite unions (which is not needed in Theorem 5.1) cannot be omitted. Indeed, if $X = \boldsymbol{I}_1 \oplus \boldsymbol{I}_2$ (the topological sum of two copies of the closed interval $\boldsymbol{I}$), if $Y = \boldsymbol{I}$, and if $f: X \to Y$ identifies $0 \in \boldsymbol{I}_1$ with $0 \in \boldsymbol{I}_2$, then f is tri-quotient (because it is perfect) but $\{\boldsymbol{I}_1, \boldsymbol{I}_2\}$ is an open cover of X for which the interiors of $f(\boldsymbol{I}_1)$ and $f(\boldsymbol{I}_2)$ do not cover Y.

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